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A Spectral Method for Nonlinear Wave Equations

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In this paper we introduce a new integration technique to find radially symmetric solutions of nonlinear wave equations, defined on a spherical domain with Dirichlet boundary conditions, and we use this technique to study orbital stability of the standing waves. We prove analytically that the wave of lowest energy is stable, and we show by numerical computations that the standing waves of higher energy are unstable. © 1987 Academic Press, Inc.

INTRODUCTION

In this paper we introduce a new integration technique to approximate radially symmetric solutions of nonlinear wave equations. Our numerical scheme is a modification of the collocation method (see [6]) which is commonly used for numerical integration of hyperbolic equations.

The standard spectral method uses Legendre polynomials or Tchebysheff polynomials to obtain the collocation points and the collocation basis. However, the weak part of this method is that the collocation points are not equidistant and often do not provide the desired information about the approximated solution in specific intervals. Attempts to find a basis which provides equidistant distribution of collocation points have failed so far. Also, because of the form of the matrices involved, the standard collocation method requires a considerable amount of computations.

We propose to use the *zeroth* Bessel function $J_0(\sqrt{\lambda_j}\, r)$, $j = 1,..., m$, $0 \leqslant r \leqslant 1$, where $\sqrt{\lambda_j}$ is the jth zero of J_0 for all j as an approximation basis, and

$$\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_{m+1}}}, ..., \frac{\sqrt{\lambda_m}}{\sqrt{\lambda_{m+1}}}$$

as collocation points. Surprisingly, these points are almost uniformly distributed with an approximate distance of 0.02 units between them.

In the radially symmetric case $J_0(\sqrt{\lambda_j}\, r)$, $j = 1,..., m$, are the eigenfunctions of the Laplace operator. Thus, in this basis the Laplace operator can be approximated by a diagonal matrix. Moreover, the transformation matrix between the spectral space and the physical space is symmetric. This reduces, to some degree, the amount of

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work required. Because of the relation between the *zeroth* and the *first* Bessel function the gradient of the solution is easily computed.

All this suggests that it is very natural to use our technique to approximate radially symmetric solutions of nonlinear wave equations with Dirichlet boundary conditions. In particular, the boundary conditions are automatically satisfied.

In this paper we apply this technique to study orbital stability of standing waves, which in itself is a very important and interesting question in mathematical physics.

We consider the nonlinear wave equation

$$u_{tt} - \Delta u + f(u) = 0$$

defined on a smoothly bounded domain $D \subset R^n$ with the Dirichlet boundary condition

$$u|_{\partial D} = 0.$$

The standing waves are solutions of the form

$$u(t, x) = e^{i\omega t}\varphi(x)$$

where φ is a real valued function satisfying the elliptic equation

$$-\Delta u - \omega^2 u + f(u) = 0$$

on D with the Dirichlet boundary condition

$$u|_{\partial D} = 0.$$

By orbital stability of a standing wave $e^{i\omega t}\varphi(x)$ we mean the stability of the set

$$S = \{e^{i\theta}\varphi \mid \theta \in R\}.$$

The ground states are the standing waves of lowest energy. In this case φ is positive (see [3]). Standing waves for which the energy is not minimal are called bound states. In this case φ is called a higher mode.

It has been shown in [7] that for any given integer $k \geqslant 0$ there exist exactly two real valued solutions of the elliptic problem with k zeros, namely φ and $-\varphi$. In the following we will discuss only those solutions φ satisfying the conditions $\varphi(0) > 0$. It is also known that if D is a unit ball in R^n, then the positive solution of the elliptic problem is radially symmetric [4].

Stability properties of the ground state of various problems have been studied by many authors (see, for example, [1, 2, 5, 9–11]). For $\omega = 0$ a blow-up theorem for the ground state of a special class of nonlinear wave equation on a bounded domain has been given by Payne and Sattinger [8] and for higher modes by this author [12]. However, for $\omega \neq 0$ the stability properties of standing waves for nonlinear wave equations on a bounded domain remained an open question.

In [11] Shatah and Strauss were able to show that for the Klein–Gordon equation with a very general nonlinearity the ground states are stable for some ω and unstable for other ω. On a bounded domain the situation is different. In this paper we prove the following general theorem:

THEOREM. *The ground state of the given equation is orbitally stable for all* $\omega \neq 0$ *and a very general nonlinearity.*

This result was used to test our algorithm. The numerical computations agreed with the theorem. It also turned out that our scheme provides good accuracy even for a relatively small number of collocation points, the invariants of the given equation are conserved, and the convergence is fast.

For bound states our technique has shown that in the two-dimensional case the bound state $e^{i\omega t}\varphi(x)$ of the given wave equation with a cubic nonlinearity is orbitally unstable, if φ is the radially symmetric solution of the elliptic equation with one zero. This result has also been obtained for the bound state, induced by the radially symmetric solution of the elliptic equation with two zeros. It seems that this result is also true for φ with any number of zeros and for more general nonlinearities.

It is also worthwhile to observe the behavior of the energy in time on different parts of the space, and the energy exchange between these parts of the space. It is expected that in case of an unbounded domain the energy will decay at infinity. Our computations show that on a bounded domain the picture is different. Such behavior of the energy seems to be characteristic for these types of boundary value problems.

Throughout this paper we employ the following notation: $\| \ \|$ is the norm in $H_0^1(D)$ defined by

$$\|u\|^2 = \int_D |\nabla u|^2 \, dx,$$

$\| \ \|_q$ is the usual norm in $L^q(D)$; by $\rightharpoonup$ we denote weak convergence and by $\to$, strong convergence.

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1. PRELIMINARIES

Consider the nonlinear wave equation

$$u_{tt} - \Delta u + f(|u|)\, u = 0 \tag{1}$$

on a smoothly bounded domain $D \subset R^n$ with the Dirichlet boundary condition

$$u|_{\partial D} = 0.$$

The standing waves of this equation are solutions of the form

$$u(t, x) = e^{i\omega t}\varphi(x),$$

where φ satisfies the elliptic equation

$$-\Delta u - \omega^2 u + f(|u|)\, u = 0 \tag{2}$$

on D with the boundary condition

$$u|_{\partial D} = 0.$$

Throughout this paper we make the following assumptions:

(i) $f \in C^1(R, R)$;

(ii) $f(z) \geqslant 0$ for all $z \geqslant 0$;

(iii) $G(z) = \int_0^z yf(y)\, dy$;

(iv) $f(z) = O(|z|^{q-2})$ for large z and $2 < q \leqslant 2n/(n-2)$.

For example, $f(|u|) = |u|^2$.

Consider the functionals E and Q on $H_0^1(D) \times L^2(D)$, describing energy and charge:

$$E(u, v) = \frac{1}{2}\int_D |v|^2\, dx + \frac{1}{2}\int_D |\nabla u|^2\, dx + \int_D G(|u|)\, dx$$

and

$$Q(u, v) = \operatorname{Im} \int_D v\bar{u}\, dx.$$

They are invariant under the flow of (1) (see [13]).

Consider the set

$$M_c = \{(u, v) \in H_0^1(D) \times L^2(D) \mid Q(u, v) = c\},$$

for $c > 0$, endowed with the subset topology. Certainly, M_c is a Finsler submanifold of $H_0^1(D) \times L^2(D)$ of codimension 1.

The following proposition states some of the properties of M, E, and Q.

Proposition 1.1. *The following statements hold for all* $c > 0$:

(i) *M_c is bounded away from zero.*

(ii) *There exists a constant $K > 0$ such that $E(u, v) \geqslant K$ for all $(u, v) \in M_c$.*

(iii) *If $(u, v) \in M_c$ is a critical point of $E|_{M_c}$, then there exists an $\omega \in R$ such that $v = i\omega u$ and u is a solution of* (2). *If u is a solution of* (2) *for some $\omega > 0$, then $(u, i\omega u)$ is a critical point of $E|_{M_c}$ for $c = \omega \int_D |u|^2\, dx$.*

(iv) *Q is compact.*

Proof. (i) Let $(u, v) \in M_c$ be arbitary. Then, using Schwartz's inequality and Sobolev's imbedding theorem, we obtain that

$$c = \operatorname{Im} \int_D v\bar{u}\, dx \leqslant \left| \int_D v\bar{u}\, dx \right| \leqslant \left(\int_D |v|^2\, dx \right)^{1/2} \left(\int_D |u|^2\, dx \right)^{1/2}$$
$$\leqslant K \left(\int_D |v|^2\, dx \right)^{1/2} \left(\int_D |\nabla u|^2\, dx \right)^{1/2},$$

where $K > 0$ is the Sobolev constant. Therefore $(u, v) \neq 0$ and

$$\|(u, v)\|^2 = \int_D |\nabla u|^2\, dx + \int_D |v|^2\, dx \geqslant 2 \left(\int_D |v|^2\, dx \right)^{1/2} \left(\int_D |\nabla u|^2\, dx \right)^{1/2} \geqslant \frac{2c}{K}.$$

(ii) Using part (i) we obtain that

$$E(u, v) \geqslant \frac{1}{2} \int_D |v|^2\, dx + \frac{1}{2} \int_D |\nabla u|^2\, dx \geqslant \frac{c}{K}.$$

for all $u \in M_c$.

(iii) If $(u, v) \in M_c$ is a critical point of $E|_{M_c}$, then there exists a Lagrange multiplier $\omega \in R$ such that

$$E'(u, v) = \omega Q'(u, v).$$

Substituting $v = i\omega u$ into the above equation, we obtain that u is a nontrivial solution of (2). The last part of the statement is obvious.

(iv) Let $\{(u_i, v_i)\}$ be a bounded sequence in $H_0^1(D) \times L^2(D)$. Then there exists a subsequence of $\{u_i\}$ and a subsequence of $\{v_i\}$, which we denote in the same way, such that $u_i \rightharpoonup u$ in $H_0^1(D)$ and $v_i \rightharpoonup v$ in $L^2(D)$. Since the imbedding $H_0^1(D) \subsetneq L^2(D)$ is compact, it follows that $u_i \to u$ strongly in $L^2(D)$. The definition of weak convergence implies that

$$\int_D v_i \bar{u}_i\, dx \to \int_D v\bar{u}\, dx,$$

and therefore, $Q(u, v) = \lim_{i \to \infty} Q(u_i, v_i)$.

This completes the proof of the proposition.

2. Existence of a Stable Standing Wave

In this section we obtain the ground state of $E|_{M_c}$ using variational methods (see, for example, [3]), and we show that the corresponding standing wave is orbitally stable.

PROPOSITION 2.1. *For any $c>0$ there exists a nontrivial solution φ of* (2) *such that*

$$E(\varphi, i\omega\varphi) = \inf_{w \in M_c} E(w). \tag{3}$$

where $\omega = c/\int_D |\varphi|^2 \, dx$.

Proof. Let $\{(u_k, v_k)\}$ be a minimizing sequence in M_c, i.e., $(u_k, v_k) \in H_0^1(D) \times L^2(D)$, $Q(u_k, v_k) = c$ for all $k \in N$ and

$$E|_{M_c}(u_k, v_k) \to \inf_{w \in M_c} E|_{M_c}(w).$$

Since $\{(u_k, v_k)\}$ is a bounded sequence in $H_0^1(D) \times L^2(D)$, we can extract weakly convergent subsequences of $\{u_k\}$ and $\{v_k\}$, which we denote by the same symbol. So, let $u_k \rightharpoonup \varphi$ in $H_0^1(D)$, and $v_k \rightharpoonup \psi$ in $L^2(D)$. Since the imbedding $H_0^1(D) \subsetneq L^q(D)$ is compact, it follows that $u_k \to \varphi$ strongly in $L^q(D)$ and, using assumption (iv), we obtain that

$$\lim_{k \to \infty} \int_D G(|u_k|) \, dx = \int_D G(|\varphi|) \, dx.$$

This and lower semicontinuity of weak limits imply that

$$E|_{M_c}(\varphi, \psi) \leqslant \liminf_{k \to \infty} E|_{M_c}(u_k, v_k) = \inf_{w \in M_c} E|_{M_c}(w).$$

On the other hand, the compactness of Q implies that

$$Q(\varphi, \psi) = \lim_{k \to \infty} Q(u_k, v_k) = c,$$

i.e., $(\varphi, \psi) \in M_c$.

Therefore,

$$E|_{M_c}(\varphi, \psi) = \inf_{w \in M_c} E|_{M_c}(w),$$

and $\{(u_k, v_k)\}$ converges strongly to (φ, ψ) in $H_0^1(D) \times L^2(D)$.

Proposition 1.1 implies that $\psi = i\omega\varphi$ and $\omega = c/\int_D |\varphi|^2 \, dx$. This completes the proof.

The set of all the solutions of the minimization problem (3) is of the form

$$S = \{e^{i\theta}\varphi \mid \theta \in R\},$$

otherwise $(|\varphi|, i\omega|\varphi|)$ would have a lower energy than $(\varphi, i\omega\varphi)$.

Thus, in order to prove orbital stability of the standing wave $e^{i\omega t}\varphi(x)$ we have to prove stability of the set S with respect to the flow of (1). This leads to the following theorem.

THEOREM 2.2. *For any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that, if*

$$\|u(0) - \varphi\|_{H_0^1(D)} + \|u_t(0) - i\omega\varphi\|_{L^2(D)} < \delta$$

then

$$\inf_{\psi \in S} (\|u(t) - \psi\|_{H_0^1(D)} + \|u_t(t) - i\omega\psi\|_{L^2(D)}) < \varepsilon$$

for all $t \in R$, and all solutions u of Eq. (1).

Proof. Assume the contrary. Then there exists an $\varepsilon > 0$, a sequence $\{(u^j(0), u_t^j(0))\}$ in $H_0^1(D) \times L^2(D)$, and a sequence $\{t_j\}$ in R such that

$$\{(u^j(0), u_t^j(0))\} \to (\varphi, i\omega\varphi)$$

in $H_0^1(D) \times L^2(D)$ as $j \to \infty$, and

$$\|u^j(t_j) - \psi\|_{H_0^1(D)} + \|u_t^j(t_j) - i\omega\psi\|_{L^2(D)} > \varepsilon$$

for all $\psi \in S$ and j big enough.

Since energy and charge are conserved under the flow of (1), it follows that

$$E(u^j(t_j), u_t^j(t_j)) \to E(\varphi, i\omega\varphi)$$

and

$$Q(u^j(t_j), u_t^j(t_j)) \to Q(\varphi, i\omega\varphi)$$

as $j \to \infty$.

This implies that $\{u^j(t_j)\}$ is a bounded sequence in $H_0^1(D)$ and, therefore, it contains a subsequence, denoted in the same way, which converges weakly to a function ψ in $H_0^1(D)$. Since the imbedding $H_0^1(D) \subsetneq L^q(D)$ is compact, it follows that $\{u^j(t_j)\}$ converges strongly to ψ in $L^q(D)$ and

$$\lim_{j \to \infty} \int_D G(|u^j(t_j)|)\, dx = \int_D G(|\psi|)\, dx.$$

For the same reason there exists a sequence $\{u_t^j(t_j)\}$ which converges weakly to a function $\tilde{\psi}$ in $L^2(D)$.

The compactness of Q implies that $Q(\psi, \tilde{\psi}) = c$, i.e., $(\psi, \tilde{\psi}) \in M_c$. Moreover, by lower semicontinuity of weak limits we obtain that

$$E(\psi, \tilde{\psi}) \leqslant \liminf_{j \to \infty} E(u^j(t_j), u_t^j(t_j)) = E(\varphi, i\omega\varphi),$$

and therefore, $E(\psi, \tilde{\psi}) = E(\varphi, i\omega\varphi)$. Thus $(\psi, \tilde{\psi}) \in S$ and $\{u^j(t_j), u_t^j(t_i)\}$ converges strongly to $(\psi, \tilde{\psi})$. This contradicts the assumption.

3. Existence of Unstable Standing Waves and Their Numerical Approximation

In this section we study radially symmetric solutions of the boundary value problem

$$u_{tt} - \Delta u + |u|^2 u = 0 \tag{4}$$

with the Dirichlet boundary condition

$$u|_{\partial D} = 0,$$

where D is the unit disk in R^2.

For the radially symmetric case the boundary value problem (4) can be written in the form

$$u_{tt} - u_{rr} - \frac{1}{r} u_r + |u|^2 u = 0 \tag{5}$$

for $0 < r < 1$ with the boundary conditions

$$u_r(0) = 0, \qquad u(1) = 0.$$

In this notation the standing waves are solutions of Eq. (5) of the form $u(t, r) = e^{i\omega t}\varphi(r)$, where φ satisfies the elliptic equation

$$-u_{rr} - \frac{1}{r} u_r - \omega^2 u + |u|^2 u = 0, \qquad 0 < r < 1, \tag{6}$$

with the boundary conditions

$$u_r(0) = 0, \qquad u(1) = 0.$$

First we outline an integration technique for the boundary value problem (5) which uses the collocation approximation (see [6]). This method is based on the following.

We choose a number of collocation points $r_1, ..., r_m$ on the interval (0, 1), and a basis $(\Phi_1, ..., \Phi_m)$ in the approximation space B such that the matrix $\{\Phi_i(r_j)\}_{i,j=1,...,m}$ is nonsingular. We define then the projection operator from $H_0^1(D)$ into B by

$$Pu(r) = \sum_{j=1}^{m} b_j \Phi_j(r),$$

where b_j are solutions of the linear equation

$$\sum_{j=1}^{m} b_j \Phi_j(r_i) = u(r_i), \qquad i = 1,..., m.$$

Thus, at the collocation points we obtain that

$$Pu(r_i) = u(r_i).$$

Since the *zeroth* Bessel function $J_0(\sqrt{\lambda}\, r)$ is the solution of the eigenvalue problem

$$u_{rr} + \frac{1}{r} u_r + \lambda u = 0$$

with the boundary conditions

$$u_r(0) = 0, \qquad u(1) = 0,$$

it seems natural to choose

$$(J_0(\sqrt{\lambda_1}\, r),..., J_0(\sqrt{\lambda_m}\, r))$$

as an approximation basis, and

$$\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_{m+1}}},..., \frac{\sqrt{\lambda_m}}{\sqrt{\lambda_{m+1}}}$$

as collocation points, where $\sqrt{\lambda_j}$ is the jth zero of J_0 for all j.

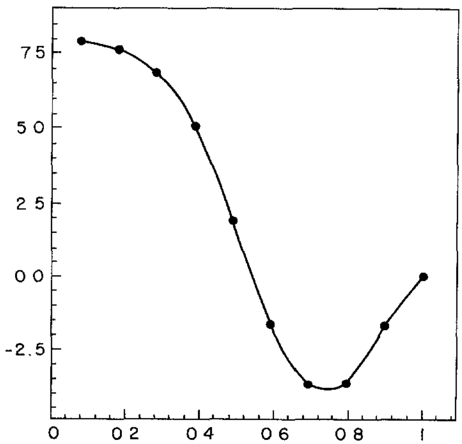

FIG. 1. Initial data versus space variable. The dots are the collocation points.

For $m=49$ the collocation points are almost equidistant, with an approximate distance of 0.02 units between them.

In Fig. 1 we can see the uniform distribution of the collocation points in the interval (0, 1) for $m=9$. For this basis we obtain that

$$u(r_i)=\sum_{j=1}^{m} b_j J_0(\sqrt{\lambda_j}\, r_i)$$

$$-\Delta u(r_i)=-u_{rr}(r_i)-\frac{1}{r}u_r(r_i)=\sum_{j=1}^{m} b_j \lambda_j J_0(\sqrt{\lambda_j}\, r_i)$$

$$\nabla u(r_i)=u_r(r_i)=-\sum_{j=1}^{m} b_j \sqrt{\lambda_j}\, J_1(\sqrt{\lambda_j}\, r_i),$$

where J_1 is the first Bessel function, and $r_i=\sqrt{\lambda_i}/\sqrt{\lambda_{m+1}}$, $i=1,\dots,m$.

Let $M=\operatorname{diag}(\lambda_1,\dots,\lambda_m)$, let

$$J=\begin{pmatrix} J_0(\sqrt{\lambda_1}\, r_1) & \cdots & J_0(\sqrt{\lambda_m}\, r_1) \\ \vdots & \ddots & \vdots \\ J_0(\sqrt{\lambda_1}\, r_m) & \cdots & J_0(\sqrt{\lambda_m}\, r_m) \end{pmatrix},$$

let $u=(u(r_1),\dots,u(r_m))$, and let $b=(b_1,\dots,b_m)$. In this notation $u=Jb$, $-\Delta u=JMJ^{-1}u$, and we obtain the integration scheme for Eq. (5),

$$\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}=-JMJ^{-1}\frac{u_{k+1}+2u_k+u_{k-1}}{4}-|u_k|^2 u_k \tag{7}$$

or, equivalently,

$$u_{k+1}=4Au_k-2u_k-u_{k-1}-Ah^2|u_k|^2 u_k,$$

where

$$A=\left[I+\frac{h^2}{4}JMJ^{-1}\right]^{-1},$$

h is the step size, and I is the identity matrix.

In the previous section we showed that the ground state is stable. Let us use this result to test our algorithm. In order to obtain the positive solution of (6), we integrate first the equation

$$v''+\frac{1}{x}v'+v-|v|^2 v=0 \tag{8}$$

with the initial conditions

$$v(0)=a, \qquad v'(0)=0$$

for some $a>0$.

TABLE I

Time	Energy	Charge
0.02	564.58	78.02
0.1	564.64	78.01
0.2	564.75	78.02
0.4	564.55	78.02
0.8	563.94	78.02
4.02	564.34	78.02
4.5	565.47	78.02
4.75	564.55	78.02

Let v be a solution of (8), and let ω be its first zero. Then $\varphi(r) = \omega v(\omega x)$ is the positive solution of (6).

Let us choose the initial data (u_0, u_1) close to $(\varphi, i\omega\varphi)$, and let us denote the computed solution of (5) by $u(t, r)$. Since $u(t, r)$ is complex valued, it has the representation

$$u(t, r) = |u(t, r)|\, e^{i\tilde{\omega}(t,r)}$$

for some real valued function $\tilde{\omega}$.

Our computation showed that, in the discrete sense, $\tilde{\omega}(t, r) \approx \omega t$ and $|u(t, r)| \approx \varphi$ at all collocation points r and for all t in the interval of observation $[0, T]$. This implies that $u(t, r) \approx \varphi(r)\, e^{i\omega t}$ and

$$\|u(t) - \varphi\|_{H_0^1(D)} + \|u_t(t) - i\omega\varphi\|_{L^2(D)} < \varepsilon(t) < \max_{t \in [0, T]} \varepsilon(t) = \varepsilon.$$

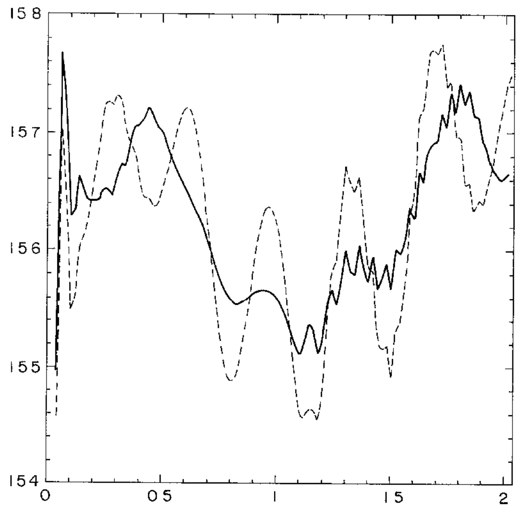

FIG. 2. The $H_0^1(D)$-norm of the computed solution vs time. The solid line denotes the solution with the initial data $u_0 = 1.001\varphi$, $u_1 = (1/1.001)\, e^{0.02\omega t}\varphi$. The dashed line denotes the solution with the initial data $u_0 = 1.0001\varphi$, $u_1 = (1.0001)\, e^{0.02\omega t}\varphi$, $\omega = 8.183884$. However, $\|\varphi\| = 1.515$.

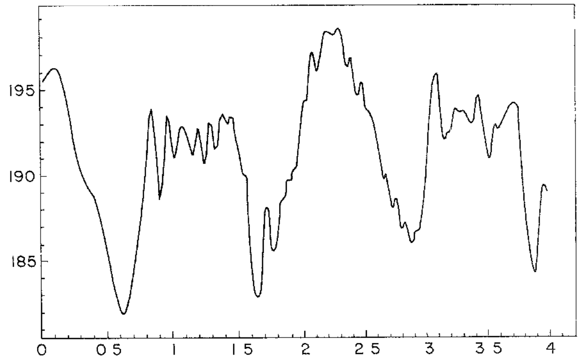

FIG. 3. Energy versus time for $0 \leqslant x \leqslant 0.357$ (first interval).

Our computation also showed that by choosing the initial data closer to $(\varphi, i\omega\varphi)$ we can make ε smaller. This implies the stability of the ground state which agrees with Theorem 2.2.

Now our goal is to apply this scheme and to give numerical evidence for the instability of the bound state, induced by the solution of (6) with one zero.

Let v be a solution of (8) with $v(0) = 9.8$ and let ω be its second zero. Then, $\varphi(r) = \omega v(\omega x)$ is a solution of (6) with one zero (see Fig. 1). So, we start with the initial data (u_0, u_1) close to $(\varphi, e^{i\omega h}\varphi)$, where $\omega = 8.183884$ and φ is the solution of (6) with one zero.

The scheme (7) conserves energy and charge, which are the invariants of problem (4) (see Table I). Also, the same accuracy has been achieved for $m = 49$ and $m = 9$.

Comparing the $H_0^1(D)$-norms of φ and the computed solution, we obtain that

$$|\, \|u(t)\| - \|\varphi\|_{H_0^1(D)}| > \varepsilon$$

for small ε. Therefore,

$$\|u(t) - e^{i\omega t}\varphi\|_{H_0^1(D)} > \varepsilon$$

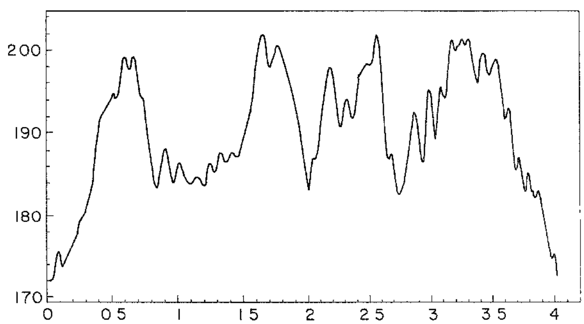

FIG. 4. Energy versus time for $0.357 \leqslant x \leqslant 0.658$ (second interval).

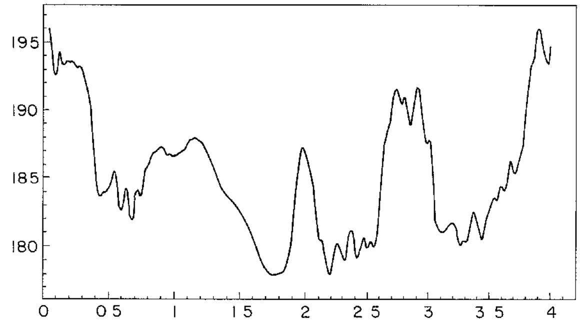

FIG. 5. Energy versus time for $0.658 \leqslant x \leqslant 1$ (third interval).

for all $t \in R$. This result holds even if we choose the initial data very close to $(\varphi, i\omega\varphi)$ (see Fig. 2). Thus, the corresponding standing wave is unstable.

Since the minimum of the energy is not achieved at φ, the solution has some excess energy. Let us divide the interval $[0, 1]$ into three parts, and let us discuss the changes of the energy in time on each of these intervals. Although the energy is conserved on $[0, 1]$, this is not the case on each interval.

Figures 3–5 show that initially the energy is equally concentrated in the first and the third intervals, and is minimal in the second one. In time, the excess energy from the first and the third intervals moves into the second one, and then back, but mostly into the first interval; after some interaction, it moves back into the second. The second interval always retains some of the energy; the maximal energy level in the first interval exceeds its initial value, while the energy level in the third interval drops much lower than it was initially. This process continues until the energy in each interval reaches its initial level.

This inelastic behavior of the energy and the interaction of the energy seem to be characteristic phenomena for nonlinear wave equations on a bounded domain.

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